

Isopin-dependent $o(4,2)$ symmetry of self-dual Wu-Yang monopoles

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July 1991

Abstract.

A spinless particle in an $SU(2)$ self-dual Wu-Yang monopole field is shown to admit an $o(4,2)$ dynamical symmetry with isospin dependent generators found previously by Barut and Bornzin. This same symmetry arises for a spinless particle with anomalous charge studied by D'Hoker and Vinet, which we relate to a (spinless) 'nucleon' in the self-dual Wu-Yang monopole's field.

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1. Introduction

In a little-known paper [1] Barut and Bornzin pointed out that the spin-dependent generators

$$\begin{aligned}
 (1.1) \quad & \mathbf{J} = \mathbf{r} \times \mathbf{p} + \frac{1}{2}\boldsymbol{\sigma} \\
 & \Gamma_0 = \frac{1}{2}\left(rp_r^2 + r + \frac{\mathbf{J}^2}{r}\right) \\
 & \Gamma_4 = \frac{1}{2}\left(rp_r^2 - r + \frac{\mathbf{J}^2}{r}\right) \\
 & D = \mathbf{r} \cdot \mathbf{p} - i \\
 & \mathbf{V} = r\mathbf{p} + \frac{1}{2}\boldsymbol{\sigma} \times \hat{\mathbf{r}} \\
 & \mathbf{A} = \frac{1}{2}r\mathbf{p}^2 - \mathbf{p}(\mathbf{r} \cdot \mathbf{p}) - \frac{\mathbf{r}}{2} + \frac{\hat{\mathbf{r}}}{8r} - \frac{1}{2}\boldsymbol{\sigma} \times \mathbf{p} - \frac{i}{4r}\boldsymbol{\sigma} \times \hat{\mathbf{r}} \\
 & \mathbf{M} = \frac{1}{2}r\mathbf{p}^2 - \mathbf{p}(\mathbf{r} \cdot \mathbf{p}) + \frac{\mathbf{r}}{2} + \frac{\hat{\mathbf{r}}}{8r} - \frac{1}{2}\boldsymbol{\sigma} \times \mathbf{p} - \frac{i}{4r}\boldsymbol{\sigma} \times \hat{\mathbf{r}}
 \end{aligned}$$

(where \mathbf{r} and \mathbf{p} are the usual position and momentum and the σ 's are the Pauli spin matrices) close into $o(4,2)$. This received little attention, mostly because no physical system with this symmetry has been exhibited so far.

The clue of their proof is the observation that the operator

$$(1.2) \quad \pi = \mathbf{p} + \frac{1}{2r}\boldsymbol{\sigma} \times \hat{\mathbf{r}}$$

behaves as momentum in a gauge background, i. e. satisfies the 'monopole' commutation relations

$$(1.3) \quad [r_i, r_j] = 0, \quad [r_i, \pi_j] = i\delta_{ij} \quad \text{and} \quad [\pi_i, \pi_j] = i\mu\epsilon_{ijk}\frac{r_k}{r^3}$$

where $\mu = -\sigma \cdot \hat{\mathbf{r}}/2$. μ (whose eigenvalues are $\pm 1/2$) commutes with Π , \mathbf{r} and with \mathbf{J} . But a previous result [2] of the same authors says that, for any half-integer μ and commutation relations (1.3), the mass 0 helicity μ representation of the conformal algebra $o(4,2)$ is

given by the generators

$$\begin{aligned}
 \mathbf{L} &= \mathbf{r} \times \boldsymbol{\pi} - \mu \hat{\mathbf{r}} \\
 \Gamma_0 &= \frac{1}{2} \left(r \pi_r^2 + r + \frac{\mathbf{L}^2}{r} \right) \\
 \Gamma_4 &= \frac{1}{2} \left(r \pi_r^2 - r + \frac{\mathbf{L}^2}{r} \right) \\
 D &= \mathbf{r} \cdot \boldsymbol{\pi} - i \\
 \mathbf{V} &= r \boldsymbol{\pi} \\
 \mathbf{A} &= \frac{1}{2} \mathbf{r} \pi^2 - \boldsymbol{\pi}(\mathbf{r} \cdot \boldsymbol{\pi}) - \frac{1}{2} \mathbf{r} + \mu \frac{\mathbf{L}}{r} + \frac{\mu^2}{2r^2} \mathbf{r} \\
 &= \frac{1}{2} (\boldsymbol{\pi} \times \mathbf{L} - \mathbf{L} \times \boldsymbol{\pi}) - \hat{\mathbf{r}} \Gamma_0 \\
 \mathbf{M} &= \frac{1}{2} \mathbf{r} \pi^2 - \boldsymbol{\pi}(\mathbf{r} \cdot \boldsymbol{\pi}) + \frac{1}{2} \mathbf{r} + \mu \frac{\mathbf{L}}{r} + \frac{\mu^2}{2r^2} \mathbf{r} \\
 &= \frac{1}{2} (\boldsymbol{\pi} \times \mathbf{L} - \mathbf{L} \times \boldsymbol{\pi}) - \hat{\mathbf{r}} \Gamma_4
 \end{aligned}
 \tag{1.4}$$

($r = |\mathbf{r}|$). Substituting (1.2) into (1.4) yields exactly (1.1).

With $\mu = q = eg$, (1.4) is a symmetry ('non-invariance') algebra of the 'MIC-Zwanziger' [3] system consisting of a charged, spin 0 particle in the combined field of a Dirac monopole and of a fine-tuned inverse-square potential,

$$H_{MZ} = \frac{1}{2} \boldsymbol{\Pi}^2 + \frac{q^2}{2R^2} - \frac{q^2}{R} + \frac{q^2}{2},
 \tag{1.5}$$

where $\boldsymbol{\Pi} = (-i\partial_{\mathbf{R}} - q\mathbf{A}_D)$, \mathbf{A}_D being the vectorpotential of a Dirac monopole of unit strength, $\text{curl } \mathbf{A}_D = \mathbf{R}/R^3$. This can be established by a suitable rescaling [2] cf. (2.4) and allows to calculate the Coulomb-type spectrum (and the S -matrix) algebraically [3]. (For $\mu = 0$ (1.4) reduces to the well-known $o(4, 2)$ -generators of the Kepler problem [4]).

We show in this Letter that the algebra (1.1) arises naturally as a symmetry for a spin 0 particle in the field of a self-dual Wu-Yang monopole [5] -but with spin replaced by *isospin*.

Another application concerns the system with anomalous charge whose Hamiltonian is

$$H_{DV} = \frac{1}{2} \left[\mathbf{P} - \left(q - \frac{1}{2} \Sigma \right) \mathbf{A}_D \right]^2 + \frac{(\lambda - q)^2 - q\Sigma + \frac{1}{4}\Sigma^2}{2R^2} + \frac{\alpha}{R},
 \tag{1.6}$$

(where $\Sigma = \sigma_3$ and $\boldsymbol{\Pi} = -i\partial_{\mathbf{R}}$) considered (without a physical interpretation) by D'Hoker and Vinet [6] who found that for $\lambda = 2q$ the spectrum becomes degenerate. Below we explain this again by a dynamical symmetry similar to (1.1).

We also consider a spin 0 isospin 1/2 particle with anomalous 'nucleon-type' charge (0 or 1) in a self-dual Wu-Yang monopole and show that its Hamiltonian only differs from (1.6) in an $o(4, 2)$ preserving term. This provides a physical interpretation to the D'Hoker-Vinet system (1.6).

2. Self-dual Wu-Yang monopoles.

A self-dual 'Wu-Yang' monopole is a Dirac monopole imbedded into $SU(2)$ gauge theory to which a suitable 'hedgehog' Higgs field has been added:

$$(2.1) \quad A = iA_i^a \frac{\sigma_a}{2} = \frac{i}{2R} \sigma \times \hat{\mathbf{R}} \quad \text{and} \quad \Phi = \Phi^a \frac{\sigma_a}{2} = \left(1 - \frac{1}{R}\right) \frac{\sigma \cdot \hat{\mathbf{R}}}{2},$$

(2.1), a self-dual ($\mathbf{D}\Phi = \mathbf{B}$) Yang-Mills-Higgs system, is in fact the large- R limit of the spherically symmetric BPS (Bogomolny-Prasad-Sommerfield) monopole [5].

Let us consider a non-relativistic, spinless particle moving in the self-dual Wu-Yang monopole field. Let us assume for simplicity that our particle is an isospin doublet. Its momentum is $\mathbf{\Pi} = -i\partial_{\mathbf{R}} - i\mathbf{A} = \mathbf{P} + \sigma \times \hat{\mathbf{R}}/2R$ cf. (1.2) so the Hamiltonian is $H_{WY} = (\mathbf{\Pi}^2 + \Phi^2)/2$.

Now $\sigma_a \hat{\Phi}_a$, the projection of isospin on the Higgs field, is the electric charge operator Q_{em} whose eigenvalues are identified with the *electric charge*. If the Higgs direction is covariantly constant, Q_{em} is conserved. This is what happens in our case: Q_{em} is

$$(2.2) \quad Q_{em} = -\frac{\sigma \cdot \hat{\mathbf{R}}}{2},$$

whose eigenvalues are $q = \pm 1/2$. The operators \mathbf{R} and $\mathbf{\Pi}$ satisfy the commutation relations (1.3) with $\mu = Q_{em}$. Inserting Q_{em} into the Hamiltonian, using $Q_{em}^2 = 1/4$, we get

$$(2.3) \quad H_{WY} = \frac{1}{2}(-i\partial - Q_{em}\mathbf{A}_D)^2 + \frac{1}{8}\left(1 - \frac{1}{R}\right)^2.$$

Diagonalizing the electric charge operator Q_{em} , the Hamiltonian (2.3) reduces to two 'MIC-Zwanziger' systems (1.5) but with *opposite* charges $q = \pm 1/2$.

On the other hand, the restriction of these generators into each of the charge sectors yields [twice] the Abelian algebra (1.4) [with opposite charges]. But the non-Abelian generators have the same commutation relations as their Abelian restrictions because Q_{em} commutes with everything, explaining why (1.1) closes into $o(4, 2)$.

Notice that \mathbf{J} , the total angular momentum of a spin 1/2 particle in a spherically symmetric potential field with no monopole, replaces here the spin 0 angular momentum \mathbf{L} containing the 'monopole' contribution $-q\hat{\mathbf{r}}$. This is another illustration of the interplay of spin and isospin [7].

Apply now the same rescaling as for MIC-Zwanziger,

$$(2.4) \quad \mathbf{r} = \sqrt{q^2 - 2H} \mathbf{R} \quad \text{and} \quad \pi = \frac{\mathbf{\Pi}}{\sqrt{q^2 - 2H}}.$$

\mathbf{r} and π satisfy the same commutation relations (1.3) as \mathbf{R} and $\mathbf{\Pi}$, and (2.4) takes H_{WY} into Γ_0 in (1.1). Notice in particular that π in (2.4) is just (1.2), explaining why Barut and Bornzin could view it as momentum in a gauge background.

We conclude that (1.1) is a symmetry for the Hamiltonian (2.3) which describes an isospin carrying, spin 0 particle in a Wu-Yang monopole field. In particular, \mathbf{A} is the rescaled Runge-Lenz vector, $\mathbf{A} = \mathbf{K}_{WY}/\sqrt{q^2 - 2H}$, with

$$(2.5) \quad \mathbf{K}_{WY} = \left\{ \mathbf{R} \mathbf{P}^2 - \mathbf{P}(\mathbf{R} \cdot \mathbf{P}) - q^2 \hat{\mathbf{R}} \right\} - \frac{i}{4R} \boldsymbol{\sigma} \times \hat{\mathbf{R}} - \frac{\boldsymbol{\sigma} \times \mathbf{P}}{2} + \frac{\hat{\mathbf{R}}}{2R} (\boldsymbol{\sigma} \cdot \boldsymbol{\ell} + 1),$$

where $\boldsymbol{\ell} = \mathbf{R} \times \mathbf{P}$ is the (not conserved) orbital angular momentum.

The same procedure as for 'Mic-Zwanziger' [2,3] shows that the spectrum is

$$(2.6) \quad E_{WY} = \frac{q^2}{2} \left(1 - \left(\frac{q}{n} \right)^2 \right), \quad n = q + 1, \dots \quad (|q| = 1/2).$$

Observe that the spectrum is the same for *both signs* of the charge. The degeneracy is therefore $2(n^2 - q^2)$, twice that in the Mic-Zwanziger case. This can also be seen by using the trick $\mathbf{\Pi}^2 = -(\partial_R)^2 + \mathbf{R} \times \mathbf{\Pi}/R^2$: the numerator of the $1/R^2$ term, $\mathbf{R} \times (-i\partial_{\mathbf{R}} - Q_{em} \mathbf{A}_D)^2 + Q_{em}^2$, is just J^2 . The hamiltonian (2.3) can thus be written as

$$(2.7) \quad H_{WY} = -\frac{1}{2}(\partial_R + \frac{1}{R})^2 + \frac{\mathbf{J}^2}{2R^2} + \frac{1}{8}(1 - \frac{2}{R}),$$

showing that H_{WY} only depends on $Q_{em}^2 = 1/4$ and not on Q_{em} itself. Eqn. (2.7) allows for a simple solution cf. [3]: the radial eigenfunctions are of the Coulomb form but with the total angular momentum j replacing the orbital angular momentum ℓ :

$$(2.8) \quad R^j e^{ikR} F\left(j + 1 - \frac{i}{4k}, j + 2, -2ikR\right),$$

where $k = \sqrt{2E - 1/4}$ and F denotes the confluent hypergeometric function.

The above procedure can also be understood in fibre-bundle terms [8]: the Hopf bundle of unit Chern class over the two-sphere (extended to the punctured plane) can also be viewed as a $U(1)$ subbundle of a trivial $SU(2)$ bundle. The Hopf connection - physically, a Dirac monopole of unit charge - extends into an $SU(2)$ connection - the Wu-Yang monopole. Matter fields are sections of associated bundles with the appropriate representation space as fiber. The algebra (1.1) has the same commutation relations as its restriction onto the Hopf bundle - where it is just the 'monopole' algebra (1.4).

3. Particle with anomalous charge.

The original idea of Yang and Mills [9] was that the proton and neutron are actually two states (with electric charge 1 and 0) of the same particle related by $SU(2)$ rotations and interacting through an $SU(2)$ gauge field. This amounts to using

$$(3.1) \quad Q_{em} = q - \frac{\sigma \cdot \hat{\mathbf{R}}}{2}$$

(with $q = 1/2$) as charge operator. The momentum is thus

$$(3.2) \quad \mathbf{\Pi} = \mathbf{P} - (q - \frac{\sigma \cdot \hat{\mathbf{R}}}{2}) \mathbf{A}$$

A spinless particle with such an anomalous charge, interacting with a self-dual Wu-Yang monopole, is therefore described by the 'nucleon' ⁽¹⁾ Hamiltonian

$$(3.3) \quad H_N = \frac{1}{2} \mathbf{\Pi}^2 + \mu^2 (1 - \frac{1}{R})^2 = \frac{1}{2} \left[\mathbf{P} - (q - \frac{\sigma \cdot \hat{\mathbf{R}}}{2}) \mathbf{A}_D \right]^2 + \frac{q^2 - q\sigma \cdot \hat{\mathbf{R}} + 1/4}{2R^2} + \frac{(q - \sigma \cdot \hat{\mathbf{R}}/2)^2}{2} \left(1 - \frac{2}{R} \right).$$

When the charge operator Q_{em} is diagonalized, (3.3) only differs from the D'Hoker-Vinet expression (1.6) in a constant and an $1/R$ term. The $\mathbf{\Pi}$'s satisfy the 'monopole' commutation relations (1.3) with $\mu = Q_{em}$. With these values, the operators in (1.4) become

$$(3.4) \quad \begin{aligned} \mathbf{J} &= \mathbf{L}_0 + \frac{\sigma}{2} = \mathbf{r} \times \boldsymbol{\pi} - q\hat{\mathbf{r}} + \frac{\sigma}{2} \\ \Gamma_0 &= \frac{1}{2} \left(r p_r^2 + r + \frac{\mathbf{J}^2}{r} \right) \\ \Gamma_4 &= \frac{1}{2} \left[r p_r^2 - r + \frac{\mathbf{J}^2}{r} \right] \\ D &= \mathbf{r} \cdot \boldsymbol{\pi} - i \\ \mathbf{V} &= r\boldsymbol{\pi} + \frac{1}{2} \sigma \times \hat{\mathbf{r}} \\ \mathbf{A} &= \left\{ \frac{1}{2} (\boldsymbol{\pi} \times \mathbf{L}_0 - \mathbf{L}_0 \times \boldsymbol{\pi}) - \hat{\mathbf{r}} \Gamma_0^0 \right\} \\ &\quad + \frac{q}{2r} \sigma - q \frac{\sigma \cdot \hat{\mathbf{r}}}{r} \hat{\mathbf{r}} - \frac{i}{4r} \sigma \times \hat{\mathbf{r}} - \frac{1}{2} \sigma \times \mathbf{p} + \frac{\hat{\mathbf{r}}}{8r} \\ \mathbf{M} &= \left\{ \frac{1}{2} (\boldsymbol{\pi} \times \mathbf{L}_0 - \mathbf{L}_0 \times \boldsymbol{\pi}) - \hat{\mathbf{r}} \Gamma_4^0 \right\} \\ &\quad + \frac{q}{2r} \sigma - q \frac{\sigma \cdot \hat{\mathbf{r}}}{r} \hat{\mathbf{r}} - \frac{i}{4r} \sigma \times \hat{\mathbf{r}} - \frac{1}{2} \sigma \times \mathbf{p} + \frac{\hat{\mathbf{r}}}{8r} \end{aligned}$$

⁽¹⁾ This terminology should not be taken too seriously: unlike real protons and nucleons, our 'nucleon' has no spin and no anomalous gyromagnetic ratio.

where now $\pi = \mathbf{p} - q\mathbf{A}_D$ as for an 'ordinary' Dirac monopole of strength q so that $\mathbf{\Pi} = \pi + \sigma \times \hat{\mathbf{r}}/2$ and we used the notations $\Gamma_0^0 = (rp_r + r + \mathbf{L}_0^2/r)/2$ and $\Gamma_4^0 = (rp_r - r + \mathbf{L}_0^2/r)/2$ for the usual meanings in (1.4). These operators close, once more, into $o(4, 2)$ and for $q = 0$ they reduce to (1.1), as they should.

In order to relate this algebra to physical situation, consider first the D'Hoker-Vinet Hamiltonian (1.6) [with $-\Sigma/2$ replaced by $-\sigma \cdot \hat{\mathbf{r}}/2$]. Rescaling once more as in (2.4) (with $H = H_{DV}$), using $\mathbf{J}^2 = \ell^2 + (q - \sigma \cdot \hat{\mathbf{r}}/2)^2$, H_{HV} (where now $\ell = \mathbf{R} \times \mathbf{\Pi}$), (1.6) is transformed into

$$(3.5) \quad \frac{1}{2} \left[rp_r^2 + r + \frac{\mathbf{J}^2}{r} \right] = \Gamma_0 = \frac{(\alpha/2)^2}{\sqrt{-2H_{DV}}}.$$

The spectrum of Γ_0 is known to be $n = j + m, m = 0, 1, \dots$. The D'Hoker-Vinet spectrum is therefore ⁽²⁾

$$(3.6) \quad E_{DV} = -\frac{(\alpha/2)^2}{n^2}, \quad n = q + 1, \dots,$$

By (3.5), (3.4) is a dynamical symmetry for the D'Hoker-Vinet system. Notice that the angular momentum operator \mathbf{J} is that of a spin 1/2 particle in a monopole field, $\mathbf{J} = \mathbf{L}_0 + \sigma/2$, and that \mathbf{A} corresponds to the Runge-Lenz vector

$$(3.7) \quad \begin{aligned} \mathbf{K}_{DV} = & \left\{ \frac{1}{2} (\pi \times \mathbf{L}_0 - \mathbf{L}_0 \times \pi) - q^2 \hat{\mathbf{R}} \right\} \\ & + \frac{q}{2R} \sigma - q \frac{\sigma \cdot \hat{\mathbf{R}}}{2R} \hat{\mathbf{R}} - \frac{i}{4R} \sigma \times \hat{\mathbf{R}} - \frac{1}{2} \sigma \times \mathbf{p} + \frac{\hat{\mathbf{R}}}{2R} (\sigma \cdot \ell + 1). \end{aligned}$$

Writing the D'Hoker-Vinet Hamiltonian as

$$(3.8) \quad H_{DV} = -\frac{1}{2} \left(\partial_R + \frac{1}{R} \right)^2 + \frac{\mathbf{J}^2}{2R^2} + \frac{\alpha}{R}$$

shows that the sign of the extra charge is irrelevant: the degeneracy of the energy level (3.6) is thus again $2(n^2 - q^2)$. The eigenfunctions are again Coulomb-type just as in (2.8).

For the 'nucleon' Hamiltonian (3.3) one rescales as $\mathbf{r} = \sqrt{\mu^2 - 2H_N} \mathbf{R}$ and $\tilde{\pi} = \mathbf{\Pi} / \sqrt{\mu^2 - 2H_N}$ to get

$$(3.9) \quad \frac{1}{2} \left[rp_r^2 + r + \frac{\mathbf{J}^2}{r} \right] = \Gamma_0 = \frac{\mu^2}{\sqrt{\mu^2 - 2H}}.$$

Observe that the $q \rightarrow -q$ symmetry is now *broken*: the l.h.s. is fine but for $q \neq 0$ μ^2 on the r.h.s. is an operator rather than a number and is non-invariant. This is also seen from writing the nucleon Hamiltonian (3.3) as

$$(3.10) \quad H_N = -\frac{1}{2} \left(\partial_R + \frac{1}{R} \right)^2 + \frac{\mathbf{J}^2}{2R^2} + \frac{(q - \sigma \cdot \hat{\mathbf{R}}/2)^2}{2} \left(1 - \frac{2}{R} \right)$$

⁽²⁾ For $\alpha = \pm 2q^2$ and adding a constant $q^2/2$ one would again get the spectrum (2.6)

where the last term is *not* $q \rightarrow -q$ invariant. For a fixed eigenvalue $q \pm 1/2$ of μ , however, $o(4, 2)$ is still a symmetry: with this constraint (3.4) simply reduces to (1.4) with $\mu = q \pm 1/2$. By eqn. (3.9), the spectra are

$$(3.11) \quad E_{\pm} = \frac{(q \pm 1/2)^2}{2} \left[1 - \left(\frac{q \pm 1/2}{n} \right)^2 \right], \quad n = q \pm 1/2 + 1, \dots$$

For each sign of the extra charge, the degeneracies are $n^2 - (q \pm 1/2)^2$, but the two spectra are clearly different. For a 'proton' ($\mu = +1$) for example, there exist bound states for all n but a 'neutron' ($\mu = 0$) has no bound states. The eigenfunctions are proportional to

$$(3.12) \quad R^j e^{ikR} F\left(j + 1 - i \frac{(q \pm 1/2)^2}{4k_{\pm}}, j + 2, -2ikR\right),$$

where $k_{\pm} = \sqrt{2E - (q \pm 1/2)^2}$. For $q \neq 0$, $1/2 \rightarrow -1/2$ is clearly no more a symmetry.

4. Discussion.

Our results are strongly reminding to what happens for 'dyons' i.e. for charged, spin $1/2$ particles with anomalous gyromagnetic ratio $g = 0$ or 4 in a 'MIC-Zwanziger' field [10,11]. The restrictions $Q_{em} = q = \text{const.}$ of the Hamiltonian (2.3) are quite similar to the $g = 0$ sector which is just the doubled MIC-Zwanziger system

$$(4.1) \quad H_0 = H_{MZ} \mathbf{1}_2,$$

so it is $o(4, 2)$ symmetric. Since both components have here the same electric charge and the spin is uncoupled, there is an obvious extra $o(3)$ due to invariance under rotations of ordinary spin. The only difference is that our system has *opposite* electric charges. It is puzzling whether our ' $q \rightarrow -q$ ' symmetry is actually part of such an extra $o(3)$.

On the other hand, the particle with $g = 4$ described

$$(4.2) \quad H_1 = H_0 - q \frac{\sigma \cdot \mathbf{R}}{R^3},$$

is the chiral superpartner of H_0 [9,10] and has therefore the same symmetry as H_0 .

Curiously, the same $o(4, 2)$ representation as here appears also for the 'Kaluza-Klein' monopole [12], where the $\pm q$ charges also have the same spectra.

It would also be interesting to find a geometric derivation for the isospin-dependent algebras (1.1) and (3.4) analogous to the one given for (1.4) [13].

Acknowledgements. We would like to thank Professor L. O'Raiheartaigh for hospitality in Dublin.

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